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Burgers Equation with Self-Similar Gaussian Initial Data: Tail Probabilities

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The statistical properties of solutions of the one-dimensional Burgers equation in the limit of vanishing viscosity are considered when the initial velocity potential is fractional Brownian motion (FBM). We establish the asymptotic power-law order for log-probability of large values, both velocity and shock (amplitude of velocity discontinuity). This confirms the conjecture of U. Frisch and his collaborators. Rigorous results for this problem were previously derived for the case of Brownian motion using Markov techniques. Our approach is based on the intrinsic properties of FBM and the theory of extreme values for Gaussian processes.

KEY WORDS: Burgers equation; fractional Brownian motion; tail probabilities.

1. INTRODUCTION

The equation

$$\begin{gathered} \partial_t u + u\,\partial_x u = \mu\,\partial_{xx} u \\ u(x, t=0) := u_0(x) = ds(x)/dx \end{gathered} \tag{1}$$

with a random potential $s(x)$, $x \in R^1$ is frequently called the 1-D model of Burgers turbulence.[3] In relation to cosmological applications,[14] there is the problem of a large-scale description of solutions u at large times. It is known that for a broad class of gaussian space-homogeneous initial data u_0, the long-time large-scale limits of a suitably rescaled solution $u(t, x)$ exist and can be described as solutions of the inviscid Burgers equation, i.e. as limit

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solutions of (1) with $\mu \to 0$. In cases that are nontrivial for applications the initial potential $s(x)$, $|s| < \infty$ of the scaling limit is defined either with the help of a Poissonian point field (x, s) (see Molchanov *et al.*)[7] or with the help of a continuous self-similar gaussian process $w_\gamma(x)$ of the following type

$$E\,|w_\gamma(x) - w_\gamma(y)|^2 = |x-y|^\gamma, \qquad 0<\gamma<2 \tag{2}$$

that is, $s(x)$ is the fractional Brownian motion $w_\gamma(x)$ (a nonrigorous result of Gurbatov *et al.*).[4]

For a continuous potential $s(x) = o(x^2)$, $x \to \infty$ the solution of (1) in the inviscid case is given by the Hopf–Cole formula[3]

$$u(x, t) = (x - a(x, t))/t \tag{3}$$

where $a(x, t)$ is the lower bound of points a at which the function $y \to s(y) + (y-x)^2/2t$ achieves its (global) minimum; briefly

$$a(x, t) = \arg\inf_a\,(s(a) + (x-a)^2/2t) \tag{4}$$

For fixed $t, a(x, t)$ is a nondecreasing left-continuous function of z.[11] Following the terminology of gas dynamics, a solution u, a point of discontinuity of u (or $a(x, t)$), and a magnitude of such a discontinuity will be called velocity, shock point, and shock, respectively. The papers[11, 13] have raised several serious issues related to the statistical properties of solutions of (3, 4) where either $s(x)$ or $ds(x)/dx$ is a random process $w_\gamma(x)$. The issues concern the fractal structure of points $\mathscr{L}_a = \{a(x, t_0), x \in R^1\}$ and the distribution of the shocks $m = \Delta u = a(x-0, t_0) - a(x+0, t_0)$. Computer simulations and some heuristic arguments suggest the following conjectures:[11, 13]

(i) The set $\mathscr{L}_a$ is such that

$$\begin{aligned} &\dim \mathscr{L}_a = \gamma/2 && \text{if} \quad s' = w_\gamma \\ &\text{card}\{\mathscr{L}_a \cap [-N, N]\} < \infty, && \text{if} \quad s = w_\gamma \quad \text{and} \quad N < \infty \end{aligned} \tag{5}$$

(ii) The distribution of shocks $F_m(x)$ has the following asymptotic behavior as $x \to 0$ or $x \to \infty$:

$$\begin{aligned} 0 < c_1 < F(x)\,x^\theta < c_2 < \infty, &\qquad x < \varepsilon \\ -\infty < c_- < \ln \bar{F}(x) \cdot x^{2\theta-2} < c_+ < 0, &\qquad x > \varepsilon^{-1} \end{aligned} \tag{6}$$

where $\bar{F}=1-F$ and $\theta=\gamma/2-1$ if $s=w_\gamma$. In the case $s'=w_\gamma$ we have to set $\theta=\gamma/2$ and to replace F and $\bar{F}$ by the average number of shocks of size $m>x$ in a fixed unit interval. These conjectures have received rigorous substantiation for the markovian case only so far, $\gamma=1$[1, 2, 10]; the lower bound of dim $\mathscr{L}_a$ for the case $s'=w_\gamma$ was found recently in the general case.[5]

Below we provide a proof of hypothesis (6) for the case $s=w_\gamma$, $\gamma\in(0,2)$. Estimates of type (6) will be derived both for the distribution of velocity u and shock m. We indicate constants $c_\pm$ that are the closest and uniformly bounded in γ for the distribution of u. Although the case $\gamma=1$ has been studied in great detail by Burgers,[3] the limiting values of $c_\pm$ are not known. The general case of γ presents certain difficulties related to the fact that the initial data are not markovian. This can be overcome by using the self-similarity of $w_\gamma(x)$ and the well-developed theory of extreme values for Gaussian processes.

2. PRELIMINARY REMARKS

The initial potential $s(x)=w_\gamma(x)$ possesses a self-similarity of the type

$$w_\gamma(\lambda x)\stackrel{d}{=}\lambda^{\gamma/2}w_\gamma(x) \tag{7}$$

where $=^d$ denotes the equality of finite-dimensional distributions. Therefore, solutions to (3, 4) have a similar property[11]: $\lambda^{1-\gamma/2}u(\lambda x,\lambda^{2-\gamma/2}t)=^d u(x,t)$ which reduces the study of a spatial statistic of $u(x,t)$ to the study of a statistic of the process $u(x)=u(x,t=1)$. It follows from

$$w_\gamma(x)-w_\gamma(x_0)\stackrel{d}{=}w_\gamma(x-x_0) \tag{8}$$

that the process $u(x)$ is space homogeneous (invariant under shifts along x). For this reason all distributions connected with a fixed point x have the same relevance to any other point.

It follows from the statistical symmetry $w_\gamma(-x)=^d w_\gamma(x)$ and the parity of $y=x^2/2$ that $u(x)$ is statistically odd function:

$$u(-x)\stackrel{d}{=}-u(x) \tag{9}$$

The following geometrical interpretation[10] is useful for describing solutions (3, 4). Consider a convex hull C_F of the function $F(a)=a^2/2+s(a)$. By (4)

$$a(x,t=1):=a(x)=\arg\inf_a\,(F(a)-xa)$$

where $F(a)$ can obviously be replaced with $C_F(a)$. Take the support line to C_F with a slope of x. The line may be identical with C_F at a single point $\hat{a}(x)$ (regular point x), or within an interval $[a_*(x), a^*(x)]$. Then $a(x)=\hat{a}(x)$ at regular points, while at shock points: $a(x-0, 1)=a_*(x)$, $a(x+0, 1)=a^*(x)$ and $m=u(x+0)-u(x-0)=a_*-a^*<0$.

In physics terms,[3] the shock interval (a_*, a^*) of the point x determines the positions of particles of mass δa and momentum $\delta s(a)$ which will be absorbed into a single particle after time $t=1$ having the position x, the mass $m=a^*-a_*$, and the momentum $I=s(a^*)-s(a_*)$.

Statement 1. Let $C_F(a)$ be the boundary of the convex hull $F=a^2/2+w_\gamma(a)$. Then those points a where $C_F(a)=F(a)$ have zero Lebesgue measure.

Proof. Following Sinai,[10] we will call a point a_0 special, if there is a vicinity of a_0 that depends on the sample ω where

$$F(a)\geqslant F(a_0)+k(\omega)(a-a_0),\ |a-a_0|<\varepsilon(\omega)$$

One has

$$\frac{w_\gamma(a)-w_\gamma(a_0)}{a-a_0}\geqslant k(\omega)-\frac{a+a_0}{2}\geqslant k(\omega)-\frac{\varepsilon(\omega)}{2}-a_0 \tag{10}$$

at a special point for all a: $a_0<a<a_0+\varepsilon(\omega)$.

However, the fractional Brownian motion $w_\gamma(x)$ has the following property[8]:

$$\liminf_{t\to 0}\frac{w_\gamma(t)}{t^{\gamma/2}\sqrt{|2\log\log t|}}=-1 \qquad \text{a.s.}$$

Therefore by $w_\gamma(t)\stackrel{d}{=}w_\gamma(t+a_0)-w_\gamma(a_0)$, we have that the left-hand side of (10) cannot be semibounded at a_0 for almost all samples. Hence any fixed point is not a special a.s. The standard argument based on the Fubini theorem demonstrates that the set of special points has zero Lebesgue measure a.s. The set A where $C_F(a)=F(a)$ is a subset of special points, hence mes $A=0$ a.s. ▌

3. TAIL PROBABILITIES FOR $u(x)$ AND SHOCK INTERVALS

Consider a solution of (3, 4) with $s(x)=w_\gamma(x)$, $\gamma\in(0,2)$. Below we estimate the probabilities of large velocities $u(x)$ and of the lengths of shock intervals (a_*, a^*) that cover a given point a. Note that the conditional distribution of the statistic $m = a^* - a_*$ given $a \in (a_*, a^*)$ and the unconditional one are different. This circumstance has not been noticed by Avellaneda and E,[1,2] consequently, their proof of (6b) in the case $\gamma = 1$ and $F = F_m$ needs some correction.

Below we denote $h = \gamma/2$ and

$$\Psi(x) = \int_x^\infty e^{-u^2/2}\, du$$

It is a known fact that

$$1 - x^{-2} < \Psi(x)\, x e^{x^2/2} \leqslant 1$$

Theorem. (i) *Distribution of* $u(x)$, F_u. F_u obeys (6) with $\theta = h - 1$ and constants that are uniform in h: $c_- = -1.2$, and $c_+ = -0.12$.

To be more exact, (a) the upper bound for $\bar{F}_u = 1 - F_u$ is

$$\bar{F}_u(v) \leqslant c_\gamma v^\alpha \Psi(v^{2-h}/2), \qquad v > v_0 \tag{11}$$

where

$$\alpha = \begin{cases} 0, & h > 1/2 \\ (2-h)(1/h - 2), & h < 1/2 \end{cases}$$

(b) The lower bound for $\bar{F}_u$ is

$$\bar{F}_u(v) \geqslant (2\pi)^{-1/2} p_\varepsilon \Psi(k_\gamma(1+\varepsilon)\, v^{2-h}/2), \qquad v > v_\varepsilon \tag{12}$$

where $p_\varepsilon \uparrow 1$ as $\varepsilon \to 0$,

$$k_\gamma = \begin{cases} 4(2-h)^{h-2}\, h^{-h}/\sigma_h, & h \leqslant 1/2 \\ 2\sqrt{2}(3-2h)^{3/2-h}\, (2-h)^{h-2}, & h \geqslant 1/2 \end{cases} \tag{13}$$

and

$$\sigma_h^2 = \Gamma(3/2 - h)/(\Gamma(1/2 + h)\, \Gamma(2 - 2h)) \tag{14}$$

(ii) *Conditional shock distribution*, $F_m(\cdot)$. One has the estimates (6) for F_m with $\theta = h - 1$ for all $\gamma = 2h \in (0, 2)$.

To be more exact,

$$\bar{F}_m(x) < 3\bar{F}_u(x/4) \tag{15}$$

$$\bar{F}_m(x) > p_\varepsilon \Psi(k_\gamma(1+\varepsilon)\, x^{2-h}/2), \qquad x > x_\varepsilon \tag{16}$$

where $p_\varepsilon \uparrow 1$ as $\varepsilon \to 0$ at $k_\varepsilon = 6/\sigma_h$.

3.1. Distribution F_u: proof of (11, 12)

This proof will proceed as a sequence of several lemmas.

Lemma 1. If $c_v = v^{2-h}/2$, then

$$\bar{F}_u(v) \leqslant P(\max_{[0,1]} w_\gamma(\tau) \geqslant c_v) \tag{17}$$

$$\bar{F}_u(v) \geqslant P(\min_{\tau \in [0,1]} w_\gamma(\tau + \varepsilon) > (1+\varepsilon)^2 c_v), \qquad \forall \varepsilon > 0 \tag{18}$$

Proof. Following Avellaneda and E,[2] one proceeds as follows. Let $u(0) < 0$, then $a(0) = -u(0) > 0$ (see (3, 4)). The event $\{a(0) > v\}$ entails the event

$$A = \{\exists a > v : w_\gamma(a) + a^2/2 < 0\}$$

Using the relations

$$w_\gamma(x) \stackrel{d}{=} w_\gamma(1/x)\, |x|^\gamma \stackrel{d}{=} v^{-\gamma/2} w_\gamma(v/x)\, |x|^\gamma$$

one can conclude that A is equivalent (in probability) to the event

$$\begin{aligned} \tilde{A} &= \{\exists x \in (0,1) : v^k w_\gamma(x)\, x^{-\gamma} + v^2 x^{-2}/2 < 0\} \\ &= \{\exists x \in (0,1) : w_\gamma(x) + v^{2-h} x^{\gamma-2}/2 < 0\} \\ &\subset \{\exists x \in (0,1) : w_\gamma(x) + c_v < 0\} \end{aligned}$$

In virtue of (9) one has $P(u(0) < 0) = P(u(0) > 0) = 1/2$. Therefore

$$\begin{aligned} P(|u(0)| > v) &= P(a(0) > v) < P(A) = P(\tilde{A}) \\ &< P\{\inf_{[0,1]} w_\gamma(x) < -c_v\} \end{aligned}$$

i.e., (17) is true.

We are going to prove (18). The event $\{a(0) > v\}$ can occur under the condition

$$B = \{\forall a \in (0, v) : w_\gamma(a) + a^2/2 > w_\gamma(kv) + \tfrac{1}{2}(kv)^2\}$$

where $k > 1$ is any fixed number. The use of (7) yields

$$\begin{aligned} B &\stackrel{d}{=} \{\forall x \in (0,1) : w_\gamma(x) - w_\gamma(k) \geqslant (k^2 - x^2)\, v^{2-h}/2\} \\ &\supset \{\forall x \in (0,1) : w_\gamma(x) - w_\gamma(k) \geqslant k^2 c_v\} \end{aligned} \tag{19}$$

Put $\varepsilon = k-1$. Since $w_\gamma(x) - w_\gamma(k) =^{d} w_\gamma(k-x)$, one gets

$$P(|u(x)| > v) \geqslant P(B) \geqslant P(\min_{\tau \in [0,1]} w_\gamma(\varepsilon+\tau) \geqslant (1+\varepsilon)^2 c_v)$$

i.e., (18) is true. ▮

Upper Bound of $\bar{F}_u$. The estimate (11) follows from (17) and from the following asymptotical result of Piterbarg and Prisyzhnyuk[9]:

$$\lim_{u \to \infty} [u^a \Psi(u)]^{-1} P\{\sup_{[0,1]} w_\gamma(x) > u\} = c_\gamma$$

where $a = (1/h - 2)_+$ and $x_+ = \frac{1}{2}(x + |x|)$.

The proof for the lower bound of $\bar{F}_u$ relies on two lemmas.

Lemma 2. Let

$$\hat{w}_\gamma(1) = E\{w_\gamma(1) \mid w_\gamma(x), x < 0\}$$

be the best prediction of $w_\gamma(1)$ based on observed $\{w_\gamma(x), x < 0\}$. Then the standard error of the prediction $\sigma_h^2 = E[w_\gamma(1) - \hat{w}_\gamma(1)]^2$ is given by (14).

Proof. The process $w_\gamma(x)$ admits of the following canonical representation on the entire R^1-axis in terms of white noise:[6]

$$w_\gamma(t) = c_\gamma \int_{-\infty}^{t} [(t-x)^{(\gamma-1)/2} - (-x)_+^{(\gamma-1)/2}] \, dw(x) \qquad (20)$$

i.e., the σ-algebras generated by $\{w_\gamma(s), s \leqslant t\}$ and $\{w(s), s \leqslant t\}$ are identical. Hence

$$w_\gamma(1) - \hat{w}_\gamma(1) = c_\gamma \int_0^1 (1-x)^{(\gamma-1)/2} \, dw(x)$$

and

$$\sigma_h^2 = c_\gamma^2 \int_0^1 (1-x)^{\gamma-1} \, dx = c_\gamma^2/\gamma$$

The final expression for σ_h^2 follows from (20) and the normalization $Ew_\gamma^2(1) = 1$. ▮

Lemma 3. If $\gamma \geqslant 1$, then one has for a continuous function φ:

$$P\{w_\gamma(t) \leqslant \varphi(t) \, |t|^{\gamma/2}, t \in (a,b)\} \geqslant P\{w_1(t) \leqslant \varphi(t) \, |t|^{1/2}, t \in (a,b)\}$$

Proof. Let us show that the correlation function of $\xi_\gamma(t) = w_\gamma(t)\,|t|^{-\gamma/2}$ increases as the parameter $\gamma \in (1, 2)$ does. Then Lemma 3 will follow from the well-known inequality of Slepian.[12] This states that, if two gaussian vectors $\{\xi_i\}$ and $\{\eta_i\}$ are such that $E\xi_i = E\eta_i$, $E\xi_i^2 = E\eta_i^2$ and $E\xi_i\xi_j \leqslant E\eta_i\eta_j$ then $P\{\xi_i < z_i, i = 1,..., n\} \leqslant P(\eta_i < z_i, i = 1,..., n\}$ for any vector $\{z_i\}$.

One has

$$\rho_\gamma(t, s) = E\xi_\gamma(t)\,\xi_\gamma(s) = \tfrac{1}{2}(a^\gamma + a^{-\gamma} - |a - a^{-1}|^\gamma)$$

where $a^2 = (t/s) > 1$. Therefore

$$2\partial_\gamma \rho_\gamma(t, s) = a^\gamma \ln a[f(a^{-2}) + (1 + a^{-2})^\gamma \ln(1 - a^{-2})^{-1}/\ln a]$$

where $f(x) = 1 - x^\gamma - (1 - x)^\gamma$, $x = a^{-2} \in (0, 1)$. At the stationary point: $x = 1/2$, $f = 1 - 2^{1-\gamma} > 0$, while at the end-points: $f(0) = f(1) = 0$. Therefore $f \geqslant 0$ and $\partial_\gamma \rho_\gamma(a) > 0$. ∎

Lower Bound of $\bar{F}_u$***: the case*** $\gamma < 1$. We shall use (18). In virtue of (8) we have

$$\min_{[0,1]} w_\gamma(\varepsilon + \tau) \overset{d}{=} w_\gamma(-\varepsilon) - \max_{[0,1]} w_\gamma(x) = w_\gamma(-\varepsilon) - M_\gamma$$

Let us decompose $w_\gamma(-\varepsilon)$ into the sum $\xi_\perp + \xi_\wedge$, where $\xi_\wedge = E\{w_\gamma(-\varepsilon) \mid w_\gamma(x),\ x > 0\}$ is the best prediction of $w_\gamma(-\varepsilon)$ based on the data $\{w_\gamma(x), x > 0\}$. In that case $\xi_\perp$ is a gaussian variable with parameters $(0, \varepsilon^h \sigma_h)$ (see Lemma 2). The variable $\xi_\perp$ is independent of $\{w_\gamma(x) x \geqslant 0\}$ and so of M_γ and $\xi_\wedge$. Hence, in virtue of (18) one has

$$\begin{aligned}
\bar{F}_u(v) &> P(w_\gamma(-\varepsilon) - M_\gamma > (1 + \varepsilon)^2 c_v) \\
&\geqslant P(\xi_\perp > (1 + \varepsilon)^2 c_v + \rho c_v, \xi_\wedge - M_\gamma > -c_v \rho) \\
&= p_\rho P(\xi_\perp > [(1 + \varepsilon)^2 + \rho]\, c_v) \\
&= p_\rho (2\pi)^{-1/2}\, \Psi([(1 + \varepsilon)^2 + \rho]\, \varepsilon^{-h} \sigma_h^{-1} c_v)
\end{aligned}$$

where $p_\rho = P(M_\gamma - \xi_\wedge < c_v \rho) \to 1, v \to \infty, \forall \rho > 0$.

Choose $\varepsilon = h/(2 - h)$, then

$$\bar{F}_u(v) > p_\rho (2\pi)^{-1/2}\, \Psi(k_\gamma (1 + \rho')\, c_v)$$

where $k_\gamma = (1 + \varepsilon)^2 \varepsilon^{-h} \sigma_h^{-1} = 4(2 - h)^{h-2} h^{-h}/\sigma_h$ and $\rho' = \rho/(1 + \varepsilon)^2$. One can make ρ' arbitrarily small by a suitable choice of ρ. Again, one can find $v_0(\rho)$ such that $p_\rho \simeq 1$ when $v > v_0(\rho)$.

The quantity k_γ is an increasing function of γ: it is easy to see that
$\sqrt{2} \leqslant k_\gamma \leqslant 16\sqrt{3}/9 \simeq 3.08$ in the interval (0,1), but it is unbounded at $\gamma = 2$, because $\sigma_2 = 0$. Hence the case $\gamma > 1$ calls for separate treatment.

Lower Bound of $\bar{F}_u$: the case $\gamma > 1$. Let us turn back to (19). Recalling Lemma 3, we have

$$\begin{aligned}\bar{F}_u(v) &\geqslant P(B) = P(w_\gamma(x) - w_\gamma(k) \geqslant (k^2 - x^2)\, c_v,\ \forall x \in (0, 1)) \\ &\geqslant P(w_1(x) - w_1(k) \geqslant c_v(k^2 - x^2)(k-x)^{(1-\gamma)/2},\ \forall x \in (0, 1))\end{aligned}$$

But,

$$w_1(x) - w_1(k) \overset{\mathrm{d}}{=} w_1(k-1) - w_1(x-1)$$

where $w_1(k-1)$ is independent of $\{w_1(x-1), x \in (0, 1)\}$. Therefore one can conclude, as before,

$$\bar{F}_u(v) \geqslant p_\rho P(\{w_1(k-1) \geqslant (\|\varphi\| + \rho)\, c_v\}$$

where

$$\begin{aligned}p_\rho &= P(w_1(x-1) \leqslant \rho c_v,\ \forall x \in (0, 1)) \\ &= P(\max_{[0,1]} w_1(x) \leqslant \rho c_v) = (2/\pi)^{1/2} \int_0^{\rho c_v} \exp(-x^2/2)\, dx \to 1,\ v \to \infty \\ \|\varphi\| &= \max_{[0,1]} \varphi(x) \\ \varphi(x) &= (k^2 - x^2)(k-x)^{(1-\gamma)/2}\end{aligned}$$

Put $k = (5-\gamma)/(4-\gamma)$; then $\|\varphi\| = \varphi((\gamma - 1)/(4-\gamma))$ and

$$\bar{F}_u(v) \geqslant p_\rho (2\pi)^{-1/2}\, \Psi((1+\rho')\, k_\gamma c_v)$$

where $\rho' = \rho/k_\gamma$, $k_\gamma = 4(6-2\gamma)^{(3-\gamma)/2}\,(4-\gamma)^{(\gamma-4)/2}$. The quantity k_γ is identical with (13). It decreases in the interval $\gamma \in (1, 2)$ from $k_1 = 16\sqrt{3}/9 \simeq 3.08$ to $k_2 = 2\sqrt{2} \simeq 2.83$. The largest value of k_γ thus occurs at $\gamma = 1$. This gives the value of c_- in (6b): $c_- < -\frac{1}{2}(k_1/2)^2 \simeq -1.185$.

3.2. Distribution F_m: Proof of (15), (16)

Lower Bound of $\bar{F}_u$. Let $a_- < 0 < a_+$ be points such that the curve

$$F(a) = \frac{a^2}{2} + w_\gamma(a), \qquad a \in I_\varepsilon$$

lies above the chord L which connects the points $(a_\pm, F(a_\pm))$, where $I_\varepsilon = (a_- + \varepsilon, a_+ - \varepsilon)$ and ε is small enough. Let us denote this event by A. Then the geometrical interpretation of the solution $u(x)$ (see Section 2) gives

$$A \subset \{a_* < a_- + \varepsilon < 0 < a_+ - \varepsilon < a^*\}$$

where (a_*, a^*) is the shock interval that contains 0. Therefore,

$$P(a^* - a_* > m = a_+ - a_- - 2\varepsilon) \geqslant P(A)$$

The event A means that

$$F(a) > F(a_-) + \frac{F(a_+) - F(a_-)}{a_+ - a_-}(a - a_-), \qquad a \in I_\varepsilon$$

or

$$w_\gamma(a) - w_\gamma(a_-) - \frac{w(a_+) - w(a_-)}{a_+ - a_-}(a - a_-) > \frac{a_-^2 - a^2}{2} + \frac{a_+ + a_-}{2}(a - a_-)$$

Using the relations

$$w_\gamma(a) - w_\gamma(a_-) \overset{d}{=} w_\gamma(a - a_-) \overset{d}{=} \tau_0^{\gamma/2} w_\gamma(\tau)$$

where $\tau = (a - a_-)/(a_+ - a_-)$ and $\tau_0 = a_+ - a_-$, we get

$$P(A) = P(w_\gamma(\tau) - w_\gamma(1)\,\tau < \tau(1 - \tau)\,c,\ \tau \in (\delta, 1 - \delta))$$

where $c = \frac{1}{2}\tau_0^{2-\gamma/2}$, $\delta = \varepsilon/\tau_0 \in (0, 1/2)$. Let

$$\xi_\wedge = E\{w_\gamma(1) \mid w_\gamma(x), x < 1 - \delta\}$$

Then, similarly to the above argument, $\xi_\perp = w_\gamma(1) - \xi_\wedge$ is independent of $\xi_\wedge$ and $\{w_\gamma(\tau), \tau < 1 - \delta\}$. Therefore,

$$P(A) > P(-\xi_\perp \tau > \tau(1 - \tau)\,c + c^{1/2}\tau, \forall \tau \in (\delta, 1 - \delta))\,P_{\delta, c}$$

where $P_{\delta, c} = P(w_\gamma(\tau) - \xi_\wedge \tau > -c^{1/2}\tau, \tau \in (\delta, 1 - \delta)) \to 1$ as $c \to \infty$.

Now recall the relation $\xi_\perp =^d \delta^h \sigma_h \xi, h = \gamma/2$, where ξ is a standard gaussian variable and σ_h is the standard error prediction of $w_\gamma(1)$ based on $\{w_\gamma(x), x < 0\}$, see (14). Then

$$P(A) \geqslant (2\pi)^{-1/2} P_{\delta, c} \Psi(\delta^{-h} \sigma_h^{-1}(1 - \delta)(1 + \rho)\,c)$$

where $\rho = c^{-1/2}(1-\delta)^{-1} \to 0$ as $c \to \infty$. Recall that $m = \tau_0(1-2\delta)$ and $c_m := \frac{1}{2} m^{2-h} = c(1-2\delta)^{2-h}$. Then we obtain the desired estimate

$$P(a^* - a_* > m) \geqslant (2\pi)^{-1/2} P_{\delta, c} \Psi(k_y(1-\delta)\, c_m)$$

where $k_y = \delta^{-h} \sigma_h^{-1}(1-\delta)(1-2\delta)^{h-2}$.

Putting $\delta = 1/3$, we get $k_y = 6/\sigma_h$.

Upper Bound of $\bar{F}_m$. Let $x_0 < 0$ be the position of a shock point and (a_*, a^*) be its shock interval containing 0: $a_* < 0 < a^*$. Consider the event $m = a^* - a_* \geqslant s$. The function $a(x)$ is non-decreasing, and so $u(x) = x - a(x) \leqslant x - a^*$ for all $x > x_0$. Consequently,

$$u(-\rho s) \leqslant -\rho s - a^* \leqslant -\rho s$$

if $x_0 < -\rho s$, where $\rho \in (0, 1)$ is a constant.

Let $x_0 > -\rho s$ and $m \geqslant s$. Then the center of the discontinuity in $u(x)$ at $x = x_0$, i.e. the point $(x_0, x_0 - (a^* + a_*)/2)$, lies between the straight lines $y = x \pm m/2$ in the interval $0 > x_0 > -\rho s$. However, in that case one has either

$$u(-\rho s) \geqslant m/2 - \rho s > s/2 - \rho s$$

or $u(0) \leqslant -m/2 \leqslant -s/2$. It follows that

$$P(m \geqslant s) \leqslant P(|u(-\rho s)| \geqslant \rho s) + P(|u(-\rho s)| \geqslant s(1/2 - \rho)) + P(|u(0)| \geqslant s/2)$$

Putting $\rho = 1/4$, one gets

$$P(m \geqslant s) \leqslant 3P(|u(0)| \geqslant s/4) \qquad (21)$$

Since the field of shock points is space homogeneous, m is the length of the shock interval that covers a given point.

We began by assuming $x_0 < 0$. By (9), the events $x_0 < 0$ and $x_0 > 0$ are equally probable. Therefore (21) is true in the general case.

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